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On Viscous Dissipation in the Incompressible Fluid Flow between Two Parallel Plates

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Abstract

This paper considers the problem of viscous dissipation in the laminar incompressible fluid flow between two parallel plates with Neumann boundary conditions. The method proposed to determine the temperature of the fluid makes use of the separation of variables. Thus the solution of the problem is obtained by series expansion about the complete eigenfunctions system of a Sturm-Liouville problem. Eigennfunctions and eigenvalues of this Sturm-Liouville problem is obtained by Galerkin's method.

Keywords: *dissipation, power law fluid, eigenfunction, Galerkin's method*

Introduction

The problem of viscous dissipation in the fluid flow has many practical applications. An example is oil products transportation through ducts; another is the polymer processing.

Now we will consider the incompressible laminar fluid flow between two infinite parallel plates. The plates are maintained at a constant temperature T_0 and the fluid flows through the plates with the same temperature. The flow is slow, thus we can neglect the heat transfer by conduction in flow direction. At the same time we will consider that the fluid density ρ, specific heat C_p and the heat transfer coefficient k are constants. The flow is related to a cartesian coordinate system, the Ox axis will be directed to the flow direction, the Oy axis is normal to the plates and the distance between plates is $2h$. A similar problem but with Dirichlet boundary conditions was considered in [1].

For the fluid velocity in the cross section we will consider the expression

$$v = v_0\left[1-\left(\frac{y}{h}\right)^N\right], \tag{1}$$

where v_0 is the maximal fluid velocity, $N=(n+1)/n$ where n is a rheological constant of the fluid. For Newtonian fluids $n=1$, for Bingham expanded fluid $n<1$ and for Bingham pseudoplastic fluid $n>1$.

Given these conditions the energy equation is [3], [9]:

$$\rho C_p v \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial y^2} + K\left(-\frac{\partial v}{\partial y}\right)^{n+1} \tag{2}$$

where K is a rheological constant of the fluid.

The aim of this article is to establish an approximate solution of equation (1), which verifies certain initial conditions and Neumann boundary conditions. For this we shall follow the method proposed in [1].

The plan of the article is: in section two we formulate the mathematical problem, section three will contain the algorithm for determination of eigenvalues and eigenfunctions (for the Sturm-Liouville problem obtained by method of separation of variables) with Galerkin's method [2], in section four we will present the approximate solution of the problem and the last section contains some numerical results.

The Mathematical Problem

We associate to equation (1) the initial condition

$$x = 0\,,\, T = T_0 \tag{3}$$

and boundary conditions

$$y = 0\,,\, \frac{\partial T}{\partial y} = 0\,,\, (x > 0) \tag{4}$$

$$y = h\,,\, \frac{\partial T}{\partial y} = 0\,,\, (x > 0)\,. \tag{5}$$

Condition (4) specifies that at the middle of the distance between plates the temperature has a maximum point and condition (5) specifies the adiabatic wall (Neumann boundary condition).

It is suitable to rewrite the equation (2) and the initial and boundary conditions (3), (4), (5) in dimensionless form. With the transformation group

$$\theta = \frac{T\text{-}T_0}{T_0}\,,\, \eta = \frac{y}{h}\,,\, \psi = \frac{kx}{\rho C_p H^2 v_0} \tag{6}$$

the equation (2) and the boundary conditions (3), (4), (5) become:

$$\left(1-\eta^N\right)\frac{\partial\theta}{\partial\psi} = \frac{\partial^2\theta}{\partial\eta^2} + N_{Br}\eta^N\,, \tag{7}$$

$$\psi = 0\,,\, \theta = 0\,, \tag{8}$$

$$\eta = 0\,,\, \frac{\partial\theta}{\partial\eta} = 0\,,\, (\psi > 0)\,, \tag{9}$$

$$\eta = 1\,,\, \frac{\partial\theta}{\partial\eta}\,,\, (\psi > 0)\,, \tag{10}$$

In equation (7) the coefficient N_{Br} is the Brinkman number [9].

It is easy to find that a particular solution of equation (7) which verifies condition (10) is:

$$\theta_1 = \frac{N_{Br}}{2N}\left(\eta^2 - \frac{1}{N+2}\eta^{N+2} + 2\psi\right) \tag{11}$$

The change of function

$$\theta = u + \theta_1 \tag{12}$$

leads to the equation

$$\left(1-\eta^N\right)\frac{\partial u}{\partial \psi} = \frac{\partial^2 u}{\partial \eta^2}. \tag{13}$$

The unknown function u will satisfy the conditions (9) and (10) and the initial condition (8) is replaced by:

$$\psi = 0\,,\, u = -\theta_1. \tag{14}$$

The type of equation (13) and boundary conditions (9) and (10) allow us to apply the method of separation of variables in order to determine function u. By this method function u is obtained under the form:

$$u(\psi,\eta) = \sum_{n=1}^{\infty} c_n \Phi_n(\eta)\exp\left(-\lambda_n^2\psi\right), \tag{15}$$

where Φ_n and λ_n are the eigenvalues and the eigenfunctions of Sturm-Liouville problem:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\eta^2} + \lambda^2\left(1-\eta^N\right)\Phi = 0\,, \tag{16}$$

$$\eta = 0\,,\, \frac{\mathrm{d}\Phi}{\mathrm{d}\eta} = 0\,;\, \eta = 1\,,\, \frac{\mathrm{d}\Phi}{\mathrm{d}\eta} = 0\ . \tag{17}$$

The Application of Galerkin's Method

For the determination of eigenfunctions and eigenvalues of Sturm-Liouville problem (16), (17) we will apply the Galerkin's method. For this we consider the operator:

$U : D(U) \subset L_2[0,1] \to L_2[0,1]$,

$$D(U) = \left\{\Phi \in C^2[0,1],\, \frac{\mathrm{d}\Phi}{\mathrm{d}\eta}(0) = 0\,,\, \frac{\mathrm{d}\Phi}{\mathrm{d}\eta}(1) = 0\right\}, \tag{18}$$

$U(\Phi) = \dfrac{\mathrm{d}^2\Phi}{\mathrm{d}\eta^2} + \lambda^2\left(1-\eta^N\right)\Phi$.

We look for the solution of Sturm-Liouville problem (16), (17) under the approximate form

$$\Phi(\eta) = \sum_{k=1}^{n} a_k \varphi_k(\eta), \tag{19}$$

where $n \in \mathbf{N}^*$ is the approximation level of function Φ and $(\varphi_k)_{k\in\mathbf{N}^*}$ is a complete system of functions in $L_2[0,1]$, functions which verify the conditions [4]

$$\frac{\mathrm{d}\varphi_k}{\mathrm{d}\bar{r}}(0)=0\,,\ \frac{\mathrm{d}\varphi_k}{\mathrm{d}\bar{r}}(0)=0\,,\ k\in\mathbf{N}^*\,. \tag{20}$$

The unknown coefficients a_k, $k=\overline{1,n}$ are determined given the conditions

$$<U(\Phi),\varphi_j>=0\,,\ j=\overline{1,n}\,, \tag{21}$$

the scalar product being considered in the space of square integrable function $L_2[0,1]$.

By applying these conditions we obtain the linear algebraic system in unknown a_k, $k=\overline{1,n}$:

$$\sum_{k=1}^{n}\left(\alpha_{kj}+\lambda^2\beta_{kj}\right)a_k=0\,,\ j=\overline{1,n}\,, \tag{22}$$

where

$$\alpha_{kj}=\int_0^1\frac{\mathrm{d}^2\varphi_\mathrm{k}}{\mathrm{d}\eta^2}\varphi_j\mathrm{d}\eta\,,\ j,k=\overline{1,n}\,, \tag{23}$$

$$\beta_{kj}=\int_0^1\left(1-\eta^N\right)\varphi_k\varphi_j\mathrm{d}\eta\,,\ j,k=\overline{1,n}\,. \tag{24}$$

Because the system (22) must have nontrivial solutions, we obtain the equation

$$\Delta_n\equiv\left|\,A+\lambda^2B\,\right|=0\,, \tag{25}$$

where A and B are the matrix $A=\left(\alpha_{kj}\right)_{k,j=\overline{1,n}}$, $B=\left(\beta_{kj}\right)_{k,j=\overline{1,n}}$. The solutions of equations (25) represent the approximate values, for the n approximation level, for the eigenvalues λ_1^2, $\lambda_2^2,\cdots,\lambda_n^2$.

The solution of equation (1) is difficult to obtain under this form. Consequently, through elementary transformations of determinant Δ_n this equation takes the form [5]:

$$\left|\,C-\lambda^2I_n\,\right|=0\,, \tag{26}$$

where I_n is the identity matrix of n order.

Unlike matrix A and B which are symmetrical, matrix C does not have this property anymore. Therefore we must adopt an adequate method for the determination of its eigenvalues [8].

In the following we will use the complete system of functions $\left(\varphi_k\right)_{k\in\mathbf{N}^*}$ in $L_2[0,1]$:

$$\varphi_k(\eta)=J_0\left(\mu_k\eta\right), \tag{27}$$

where J_0 is the Bessel function of the first kind and zero order and μ_k, $k\in\mathbf{N}^*$ are the roots of equation:

$$J_1(\mu)=0\,. \tag{28}$$

The integrals which appear in the formulae (23), (24) are calculated with a quadrature formula that must be compatible with Galerkin's method [6]. The eigenvalues of the Sturm-Liouville problem obtained by this method are presented in the next section.

The eigenfunctions of the problem (18), (19) have the analytical form

$$\Phi_i(\eta) = \sum_{j=1}^{n} c_{ij} J_0(\mu_j \eta), \; i = \overline{1,n} \tag{29}$$

where $(c_{i1}, c_{i2}, \cdots, c_{in}), i = \overline{1,n}$ are the eigenvectors of the matrix $A + \lambda^2 B$.

The Approximate Solution of the Problem

The unknown function u, for the n level of approximation of Galerkin's method, is obtained from (15) and (27):

$$u(\psi,\eta) = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} c_i c_{ik} \mathrm{e}^{-\lambda_i^2 \psi} \right) J_0(\mu_k \eta), \tag{30}$$

The coefficients c_i , $i = \overline{1,n}$ from (30) are determined by the use of the condition (14) and by considering that the solutions $\Phi_i, i = \overline{1,n}$ of the problem (16), (17) are orthogonal with weight $1-\eta^N$ by [0,1] [4]. Because the functions $\Phi_i, i = \overline{1,n}$ are not obtained exactly, we prefer to use the orthogonality with weight η of Bessel functions on [0,1]. Thus, for the n level of approximation, the constants c_i , $i = \overline{1,n}$ are determined by the resolution of the linear algebraic system:

$$\sum_{i=1}^{n} c_{ik} c_i = -\frac{N_{Br}}{2N} \frac{\int_0^1 \left(\eta^2 - \frac{2}{N+2} \eta^{N+2} \right) \eta J_0(\mu_k \eta) \mathrm{d}\eta}{\int_0^1 \eta J_0^2(\mu_k \eta) \mathrm{d}\eta}, \; k = \overline{1,n}, \tag{31}$$

The final solution of the problem is obtained now by using the relations (12), (15) and (30):

$$\theta(\psi,\eta) = \frac{N_{Br}}{2N} \left(\eta^2 - \frac{2}{N+2} \eta^{N+2} + 2\psi \right) + \sum_{k=1}^{n} \left(\sum_{i=1}^{n} c_i c_{ik} \mathrm{e}^{-\lambda_i^2 \psi} \right) J_0(\mu_k \eta), \tag{32}$$

Table 1. Eigenvalues of Sturm-Liouville problem

N									
0,35	0,5	0,6	0,7	0,75	0,8	0,9	1,0	1,1	1,2
λ_n^2									
0	0	0	0	0	0	0	0	0	0
14.636	15.828	16.480	17.046	17.302	17.544	17.985	18.380	18.735	19.056
55.720	59.818	62.112	64.129	65.050	65.918	67.516	68.952	70.249	71.426
122.929	131.753	136.708	141.077	143.073	144.958	148.429	151.551	154.375	156.942
216.230	231.599	240.240	247.864	251.350	254.642	260.706	266.164	271.102	275.592
335.614	359.347	372.699	384.485	389.874	394.964	404.343	412.786	420.426	427.373

481.077	514.993	534.081	550.933	558.640	565.920	579.335	591.413	602.344	612.284
652.615	698.533	724.384	747.209	757.648	767.510	785.683	802.045	816.855	830.323
850.227	909.967	943.606	973.310	986.896	999.731	1023.38	1044.68	1063.95	1081.48
1073.91	1149.29	1191.74	1229.23	1246.38	1262.58	1292.43	1319.31	1343.65	1365.78

An Application

As an example we will consider a fluid with unit Brinkman number. The eigenvalues of Sturm-Liouville problem (16), (17) are presented in table 1. The coefficients given by (23) and (24) are obtained by a numerical quadrature procedure [8]. The eigenvalues have been obtained by using the procedures BALANC, ELMHES, HQR [8]. System (31) has been solved using a procedure based on Gauss method [8].

The variation of dimensionless temperature θ given by (32) is presented in figures 1-6. In abscisse axis is the reduced transverse distance η and in axis of ordinates the dimensionless temperature θ is presented. The variation of dimensionless temperature θ is presented for some values of dimensionless variable ψ .

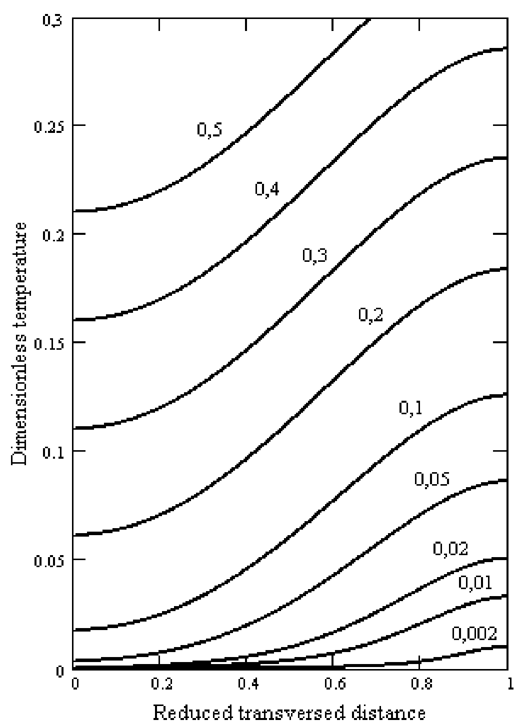

Fig. 1. Dimensionless temperature profiles for adiabatic walls, *n*=1 (Newtonian fluid), *NBr*=1

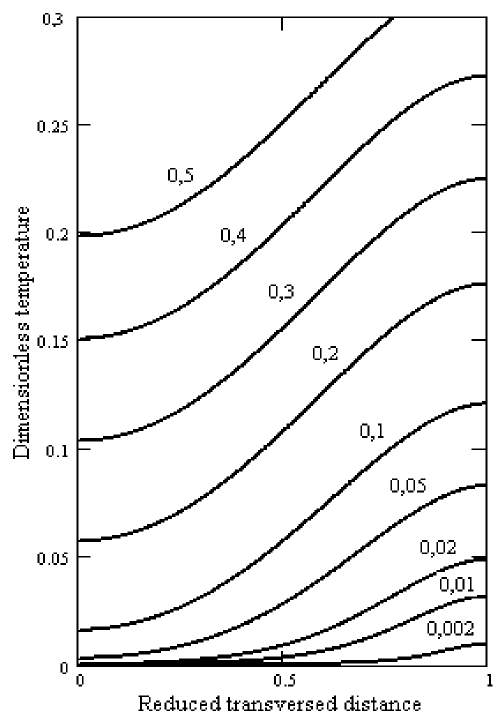

Fig. 2. Dimensionless temperature profiles for adiabatic walls, *n*=0,9, *NBr*=1

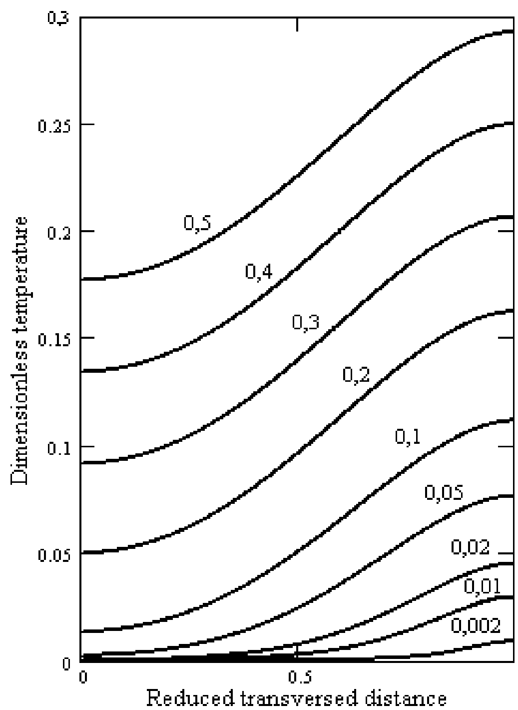

Fig. 3. Dimensionless temperature profiles for adiabatic walls, n=0,75, NBr=1

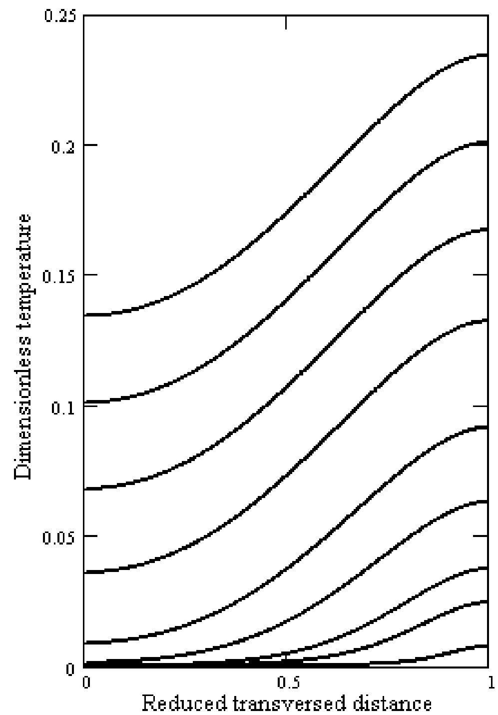

Fig. 4. Dimensionless temperature profiles for adiabatic walls, n=0,5, NBr=1

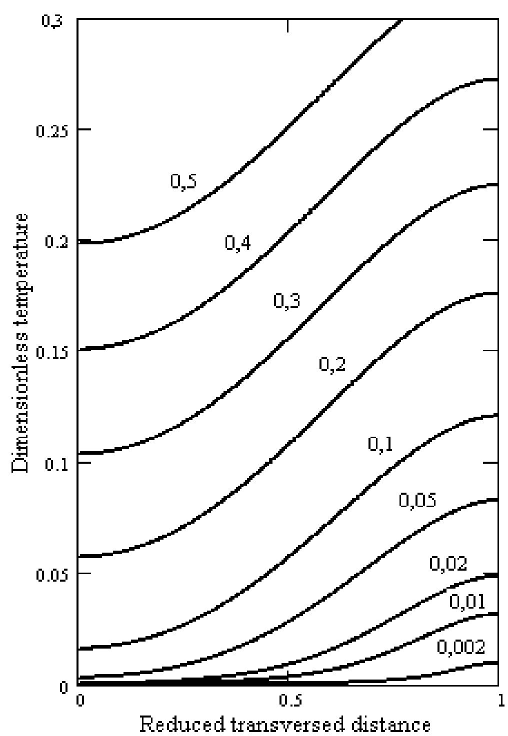

Fig. 5. Dimensionless temperature profiles for adiabatic walls, n=0,35, NBr=1

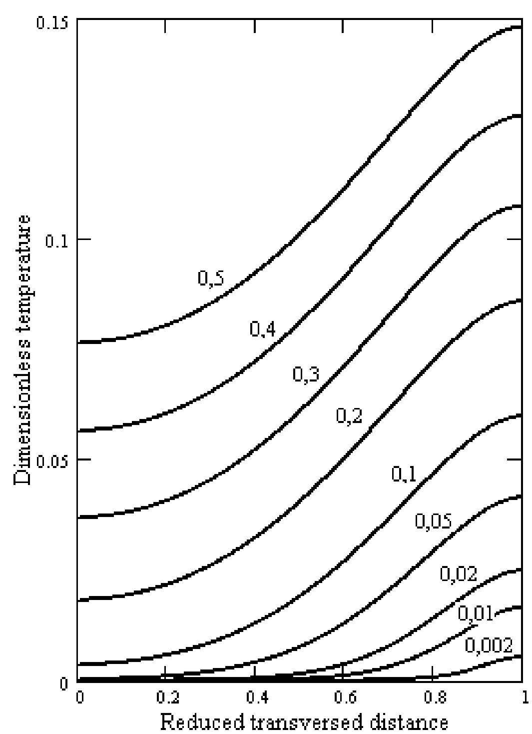

Fig. 6. Dimensionless temperature profiles for adiabatic walls, n=0,25, NBr=1

Given the results obtained, we can deduce that for a certain value of the rheological coefficient n, the temperature of the fluid is increased along the plates. For a given value of the dimensionless variable ψ, the temperature of the fluid is increased together with *n*. The results obtained in this article properly fit the results obtained in [9].

The calculations have been realized for the approximation level $n = 10$ and the algorithm presents considerable stability.

As compared to the method used by Ybarra and Eckert [7], this paper presents the advantage of a simpler algorithm which can also be adapted to other boundary conditions (Dirichlet [1] and Robin type conditions) by an appropriate changing of the condition.(17) and of the equation.(28).

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Asupra disipaţiei vâscoase în mişcarea fluidelor incompresibile printre două plăci plane paralele

Rezumat

În acest articol este studiată problema disipaţiei vâscoase în mişcarea laminară incompresibilă a unui fluid vâscos printre două plăci plane paralele cu condiţii la limită de tip Neumann. Se utilizează pentru determinarea temperaturii fluidului metoda separării variabilelor. Soluţia problemei se obţine astfel sub forma unei serii după sistemul complet de funcţii proprii unei probleme de tip Sturm-Liouville. Funcţiile şi valorile proprii ale acestei probleme Sturm-Liouville sunt obţinute cu ajutorul metodei lui Galerkin.